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On Lamé's equation of a particular kind

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Abstract
It is shown that Lamé's equation $\frac{d^2}{dz^2}X + \kappa^2 \, \mathrm{cn}^2(z, \frac{1}{\sqrt{2}})X = 0$ can be reduced to a hyper-geometric equation. The characteristic exponents of this equation are expressed in terms of elementary functions of the parameter κ. An analytical condition for parametric amplification is obtained.

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1. Introduction

Lamé's equation

$$\frac{d^2}{dz^2} X + \left(R_1 + R_2 \, \mathrm{sn}^2(z, K)\right) X = 0 \tag{1}$$

where $\mathrm{sn}(z, K)$ is the standard Jacobian elliptic sine function of modulus K[1], and R_1 and R_2 are numerical parameters, is common in many branches of mathematical physics.

Attention to this equation has been called recently by the fact that it plays an important role in certain problems of particle physics and cosmology of the Very Early Universe. In particular, it was shown [1] that in the theory of two fields ϕ and χ with potential

$$V(\phi, \chi) = \frac{\lambda\phi^4}{4} + \frac{g^2\phi^2\chi^2}{2}$$

the equation of motion of the field χ in the Heisenberg representation can be reduced to Lamé's equation under certain assumptions. More concretely, for the Minkowski space–time in the linear approximation for the field χ, it is found that after expansion of this field

[1] The elliptic sine function $\mathrm{sn}(u, K)$ and the elliptic cosine function $\mathrm{cn}(u, K)$ are defined with the help of the elliptic integral $u = \int_0^{\phi} \frac{d\alpha}{\sqrt{1-K^2 \sin^2(\alpha)}}$. Then, $\mathrm{sn}(u, K) = \sin(\phi)$ and $\mathrm{cn}(u, K) = \cos(\phi)$.

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over the eigenfunctions of the Laplace operator, the equation of motion for an eigenfunction corresponding to a particular wavenumber has the form

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2} X + \left(k^2 + \frac{g^2}{\lambda} \mathrm{cn}^2 \left(z, \frac{1}{\sqrt{2}}\right)\right) X = 0 \tag{2}$$

where $\mathrm{cn}(z, \frac{1}{\sqrt{2}})$ is the Jacobian elliptic cosine function of modulus $K = \frac{1}{\sqrt{2}}$, the parameter k is proportional to the wave number and the 'time' z is proportional to the ordinary Minkowski time coordinate. In fact, the same equation is valid in the expanding Universe in a certain regime provided that the eigenfunctions are rescaled in a proper way and the coordinate z plays the role of so-called conformal time.

Along the real axis, $\mathrm{cn}(z, \frac{1}{\sqrt{2}})$ is a periodic function with period $T = 4\mathbf{K}(\frac{1}{\sqrt{2}}) = \frac{\Gamma^2(1/4)}{\sqrt{\pi}} \approx 7.416$, where $\mathbf{K}(K)$ is the complete elliptic integral of modulus K of first type and $\Gamma(x)$ is the gamma function. It follows from the general theorem that equation (2) must contain solutions in the form $X_{1,2} = \mathrm{e}^{\mu_{1,2} z} P(z)$, where $P(z)$ is a periodic function of the period T and the coefficients $\mu_{1,2}$ are called characteristic exponents. If one of these coefficients is real and positive, the corresponding solution describes an exponential growth of the amplitude of the eigenfunction X. In modern theories of matter creation in the Universe [2–4] this growth is interpreted as a production of 'particles' of the field χ. The rate of production of the 'particles' is determined by the values of characteristic exponents and therefore the calculation of these exponents is very important for such theories. Usually the calculation of characteristic exponents is performed by numerical means and only a few cases are known with analytical solutions. In particular, the characteristic exponents have been calculated analytically in the paper [1] for the case $\frac{g^2}{\lambda} = \frac{n(n+1)}{2}$ (n is an integer).

In this note we would like to point out that the special case of equation (2) with $k = 0$,

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2} X + \kappa^2 \, \mathrm{cn}^2\left(z, \frac{1}{\sqrt{2}}\right) X = 0 \tag{3}$$

($\kappa = \frac{g^2}{\lambda}$) can be reduced to the hyper-geometric equation. This allows for exact calculation of the characteristic exponents for the important case[2], representing them in a remarkably simple form (see equations (21) and (25)).

2. Reduction of the equation to the hyper-geometric equation and the characteristic exponents

Let us consider the equation of form (3) and make the following change in the independent variable:

$$y = \mathrm{cn}^4\left(z, \tfrac{1}{\sqrt{2}}\right). \tag{4}$$

Then, we use the well-known relation

$$\frac{\mathrm{d}}{\mathrm{d}z}\mathrm{cn}(z, K) = \sqrt{(1 - \mathrm{cn}^2(z, K))(K'^2 + K^2\mathrm{cn}^2(z, K))} \tag{5}$$

where$K' = \sqrt{(1 - K^2)}$. Taking into account that in our case$K' = K = \frac{1}{\sqrt{2}}$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}z} = \pm 2\sqrt{2} y^{3/4} \sqrt{(1 - y)} \frac{\mathrm{d}}{\mathrm{d}y} \tag{6}$$

[2] This corresponds to a long-wave approximation from the viewpoint of particle physics and cosmology.

where the +sign (−sign) should be taken if y increases (decreases) with z. Also, we have

$$\frac{\mathrm{d}^2}{\mathrm{d}z^2} = 8y^{1/2}\left\{y(1-y)\frac{\mathrm{d}^2}{\mathrm{d}y^2} + \left(\frac{3}{4} - \frac{5}{4}y\right)\frac{\mathrm{d}}{\mathrm{d}y}\right\}. \tag{7}$$

Substituting the last operator into equation (3), we see that the factor $y^{1/2}$ cancels and that this equation is transformed to the standard hyper-geometric form

$$y(1-y)\frac{\mathrm{d}^2}{\mathrm{d}y^2}X + (\gamma - (\alpha+\beta+1)y)\frac{\mathrm{d}}{\mathrm{d}y}X - \alpha\beta X = 0 \tag{8}$$

where $\gamma = \frac{3}{4}$ and $\alpha, \beta = \frac{1}{8}(1 \pm \sqrt{(1+8\kappa^2)})$. The general solution to equation (8) can be written in terms of two elementary solutions valid in the vicinity of the singular point $y = 0$,

$$X = c_1\phi_1 + c_2\phi_2 \tag{9}$$

where

$$\phi_1 = F(\alpha, \beta, \gamma; y) \tag{10}$$

is the Gaussian hyper-geometric function, and

$$\phi_2 = y^{1-\gamma}F(\alpha+1-\gamma, \beta+1-\gamma, 2-\gamma; y). \tag{11}$$

Alternatively, the solution to equation (8) can be written in terms of two elementary solutions valid in the vicinity of the singular point $y = 1$,

$$X = c_3\phi_3 + c_4\phi_4 \tag{12}$$

where

$$\phi_3 = F(\alpha, \beta, \alpha+\beta+1-\gamma; 1-y) \tag{13}$$

and

$$\phi_4 = (1-y)^{\gamma-\alpha-\beta}F(\gamma-\alpha, \gamma-\beta, \gamma+1-\alpha-\beta; 1-y). \tag{14}$$

The pairs of solutions ϕ_1, ϕ_2 and ϕ_3, ϕ_4 are connected by the well-known relations (e.g. [5])

$$\phi_1 = A\phi_3 + B\phi_4 \qquad \phi_2 = C\phi_3 + D\phi_4 \tag{15}$$

$$A = \frac{\Gamma(\gamma)\Gamma(\gamma-\delta)}{\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)} \qquad B = \frac{\Gamma(\gamma)\Gamma(\delta-\gamma)}{\Gamma(\alpha)\Gamma(\beta)} \tag{16}$$

$$C = \frac{\Gamma(2-\gamma)\Gamma(\gamma-\delta)}{\Gamma(1-\alpha)\Gamma(1-\beta)} \qquad D = \frac{\Gamma(2-\gamma)\Gamma(\delta-\gamma)}{\Gamma(\alpha+1-\gamma)\Gamma(\beta+1-\gamma)} \tag{17}$$

where $\Gamma(x)$ is the Gamma function, and $\delta = \alpha + \beta$.

Let us point out that the transformation (4) is singular at the points where the function y is equal to zero or unity ($\frac{\mathrm{d}y}{\mathrm{d}z} \to 0$ when $y \to 0, 1$). A simple analysis shows that the coefficient c_2 must change its sign at the points z_i defined by $y(z_i) = 0$ and the coefficient c_4 must change its sign at the points z_j satisfying $y(z_j) = 1$. To show that, let us consider the behaviour of the function X near the points z_i. Near these points, we can approximately write

$$X \approx c_1 + c_2 y^{1/4} \tag{18}$$

and taking into account equation (6), we have

$$\frac{\mathrm{d}X}{\mathrm{d}z} \approx -\frac{c_2}{\sqrt{2}} \tag{19}$$

provided $\frac{dy}{dz} < 0$, and

$$\frac{dX}{dz} \approx \frac{c_2}{\sqrt{2}} \tag{20}$$

provided $\frac{dy}{dz} > 0$. The function X and its derivative with respect to 'time' z must be continuous functions of z. Therefore, the coefficient c_2 must change its sign at the points z_i. Similar arguments can be used to prove that the coefficient c_4 must change its sign at the points z_j where $y(z_j) = 1$.

Thus z changed over half a period of $\mathrm{cn}(z, \frac{1}{\sqrt{2}})$, a new decomposition of the solution of equation (8) should be made,

$$X = \tilde{c}_1\phi_1 + \tilde{c}_2\phi_2 \tag{21}$$

where in general the coefficients $\tilde{c}_{1,2}$ do not coincide with the coefficients $c_{1,2}$. The rules for the changing of these coefficients follow directly from the arguments mentioned above if one uses equations (9–15) taking into account the explicit form of the connection coefficients (16, 17). It is straightforward to obtain the relation

$$\tilde{c}^i = t^i_j c^j \tag{22}$$

where the components of the matrix t^i_j have the following explicit form:

$$t^1_1 = t^2_2 = \sqrt{2}\ \cos\pi(\alpha - \beta) \tag{23}$$

$$t^1_2 = \frac{8\pi\Gamma^2(2-\gamma)}{\Gamma(1-\alpha)\Gamma(1-\beta)\Gamma(1+\alpha-\gamma)\Gamma(1+\beta-\gamma)} \tag{24}$$

$$t^2_1 = \frac{8\pi\Gamma^2(\gamma)}{\Gamma(\alpha)\Gamma(\beta)\Gamma(\gamma-\alpha)\Gamma(\gamma-\beta)}. \tag{25}$$

The eigenvalues of the matrix t^i_j are

$$\lambda_{1,2} = \sqrt{2}\cos\ \pi(\alpha-\beta) \pm \sqrt{\cos 2\pi(\alpha-\beta)} \tag{26}$$

and $\alpha - \beta = \frac{\sqrt{1+8\kappa^2}}{4}$.[3]

Obviously, the multiplicators $\rho_{1,2}$ are equal to $\lambda^2_{1,2}$ and the characteristic exponents are

$$\mu_{1,2} = \frac{1}{T}\ \ln\rho_{1,2} = \frac{2}{T}\ \ln(\sqrt{2}\ \cos\pi(\alpha-\beta) \pm \sqrt{\cos 2\pi(\alpha-\beta)}). \tag{27}$$

In actual applications it is very important to know under which conditions a particular solution to equation (3) experiences parametric amplification; that is, under which conditions its amplitude increases when the 'time' z changes over the period of $\mathrm{cn}(z, \frac{1}{\sqrt{2}})$. To characterize the parametric amplification, we introduce the real quantity

$$\tilde{\mu}(\kappa) = \mathrm{Max}(\mathrm{Re}(\mu_{1,2})). \tag{28}$$

Obviously, $\tilde{\mu}(\kappa) > 0$ is the condition for parametric amplification. It is easy to see that it is satisfied when $\lambda_{1,2}$ are real; that is, when

$$n - \frac{1}{4} < \alpha - \beta < n + \frac{1}{4} \tag{29}$$

where n is an integer $\geqslant 1$. In terms of the parameter κ, these inequalities can be rewritten as

$$\sqrt{n(2n-1)} < \kappa < \sqrt{n(2n+1)}. \tag{30}$$

[3] To obtain the eigenvalues $\lambda_{1,2}$, we use the well known relations $\Gamma(x+1) = x\Gamma(x)$ and $\Gamma(1-x)\Gamma(x) = \frac{\pi}{\sin(\pi x)}$.

The expression for the quantity $\tilde{\mu}$ follows from equation (21); the definitions of the parameters α and β and the expression for the period T:

$$\tilde{\mu}(\kappa) = \frac{2\sqrt{\pi}}{\Gamma^2\left(\frac{1}{4}\right)} \ln \left\{ \sqrt{2} \left| \cos\left(\frac{\pi\sqrt{1+8\kappa^2}}{4} \right) \right| + \sqrt{\cos\left(\frac{\pi\sqrt{1+8\kappa^2}}{2} \right)} \right\}. \quad (31)$$

The quantity $\tilde{\mu}$ attains its maximal value at

$$\kappa = \sqrt{2n^2 - \tfrac{1}{8}} \quad (32)$$

where its value is

$$\tilde{\mu}_{\max} = \frac{2\sqrt{\pi}}{\Gamma^2\left(\frac{1}{4}\right)} \ln(1+\sqrt{2}) \approx 0.2377. \quad (33)$$

Note that the same value has been obtained in [1] in the asymptotic limit $\kappa \to \infty$.

3. Discussion

We were not able to find the simple formulae derived in the standard reference books on the Lamé's equation [6–8]. However, a very similar transformation between another equation of Lamé's type and a hyper-geometric equation has been discussed recently by Clarkson and Olver [9] (see also [10]). They show that the hyper-geometric equation

$$t(1-t)\frac{\mathrm{d}^2}{\mathrm{d}t^2}\Psi + \left(\frac{1}{2} - \frac{7}{6}t\right)\frac{\mathrm{d}}{\mathrm{d}t}\Psi - \sigma\Psi = 0 \quad (34)$$

and Lamé's equation

$$\frac{\mathrm{d}^2}{\mathrm{d}u^2}\Psi + 36\sigma W(u)\Psi = 0 \quad (35)$$

where $W(u)$ is the Weierstrass elliptic function with parameters $g_2 = 0$, $g_3 = \frac{1}{3^{3/2}16}$, are related to each other by the transformation

$$\mathrm{cn}(u, K) = \frac{\sqrt{3} - 1 + (1-t)^{1/3}}{\sqrt{3} + 1 - (1-t)^{1/3}} \quad (36)$$

where $K = \sqrt{\frac{1}{2} + \frac{\sqrt{3}}{4}}$. Obviously, the characteristic exponents of equation (29) can be obtained by a method similar to that described above. In general, it would be very interesting to find a general solution to the following problem: under what condition can Lamé's equation of the general form be transformed to a hyper-geometrical equation? The solution could be applied to many problems of modern particle physics and cosmology.

Finally, it is interesting to note that, in principle, the same line of argument could be applied to Lamé's equation of the general form. It is well known that after the change of variable $y_1 = \mathrm{cn}^2(z, K)$ Lamé's equation is reduced to a particular form of Heun's equation. Then the calculation of the characteristic exponents is reduced to solving of the connection problem for Heun's equation between the elementary solutions corresponding to the singular points $y_1 = 0$ and $y_1 = 1$ (e.g. [11]). This problem can indeed be solved [11], but the connection coefficients analogous to coefficients (16, 17) are now expressed in terms of a complicated series which looks rather difficult for analytic treatment of the general case.

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